Designing Flexible Manipulators With the Lowest Natural Frequency Nearly Independent of Position

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Abstract—Ideally, one would design a manipulator to be as rigid as possible. However there may be some occasions when the manipulator will be flexible. This will occur when the manipulator is very long and lightweight.

Traditional flexible manipulators are difficult to control. This is because the manipulator has a nonlinear position variant behavior. Unfortunately using a large gear reduction does not reduce the nonlinearities in the flexible modes as it does in rigid manipulators. As a result, high performance controllers will inevitably be nonlinear themselves. To make matters worse, the parameters of the system are difficult to estimate so the control needs to be robust as well.

This paper makes the point that rather than designing a controller that handles a complex system, one might build a flexible manipulator that exhibits simpler dynamic behavior than current designs. Specifically, the paper demonstrates how one can design a two dimensional flexible manipulator so that its fundamental vibration frequency is nearly independent of the rigid body position of the links. Analysis demonstrates the fundamental concept involved and experimental data verifies the analysis. What remains to be shown in future work are two points:

1) that a simple robust controller can damp the vibration of the mechanism;
2) that the extra complexity introduced by the new mechanism produces no overall negative effect.

Index Terms—Constant inertia, design, frequency invariance, vibration.

I. INTRODUCTION

Historically, mechanisms have achieved quickness and accuracy through structural rigidity. Given a volume of material, one can shape the members wisely to increase stiffness. Ultimately rigidity comes by adding mass in wise locations. This increases the weight and inertia of the mechanism that demands higher power for operation at a given speed. There is sometimes a desire to limit the volume of material used in the device. For example, in terrestrial applications high speed motion with reasonably sized motors requires a reduced mass. Another example is in space applications where one must put the mass into orbit. However, reducing mass results in flexible members that can vibrate during operation. The literature points out that this vibration can arise in the joint

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a feedback signal is lost. Despite these complications, active damping can produce impressive results.

Some recent research utilizes a linear control input which is a continuous, function of the dynamic state of the manipulator. These experiments have been conducted on a manipulator with a single flexible link [8], [9]–[11]. The dynamic state of the link is the sum of rigid body motion, which is directly measured, and vibrational motion, which is often estimated. Control of a single flexible link requires a high degree of accuracy in modeling manipulator dynamics and vibration estimation [11].

Introduction of multiple links compounds the difficulties in dynamic modeling and vibration estimation. Furthermore, the links are dynamically coupled such that the position of the outer links significantly affect the vibrational modes of the inner links. This coupling can be significant enough that control becomes difficult.

Asada [12], Yang [13], Youcef-Toumi [14], and Park [15] proposed design rules for building rigid robot manipulators with a higher dynamic performance. The design guidelines demonstrate how to build rigid manipulators with decoupled, and/or position invariant dynamics.

This paper discusses the possibility for the flexible manipulator’s design to affect the dynamic coupling. In turn, this possibility provides the potential for choosing a design through which the dynamic coupling is eliminated completely.

If decoupling can be achieved, it may become possible to control multi-link manipulators with techniques applicable to single flexible link mechanisms. The control problem is still significant but much less difficult than the control of multi-link mechanisms.

A common factor with the previous work on design rules is that they apply only to rigid manipulators. This paper discusses design rules for flexible manipulators. The paper first defines the structural dynamic problem of conventional flexible robot manipulators. Then we discuss a method for designing a flexible manipulator with a simpler dynamic response. Finally, we compute and experimentally verify the variation in eigenvalues caused by manipulator position changes in the new design.

II. EQUATIONS OF MOTION OF CONVENTIONAL 2-LINK FLEXIBLE ROBOTS

Consider the dynamic behavior of the two link planar manipulator with a flexible inner link and a rigid outer link that Fig. 1 shows. The inner link is numbered 1, the outer link is 2. The length of the inner and outer links are $l_1$ and $l_2$ respectively. Let both the inner and outer links have their mass lumped at the link’s end. The motor for the outer link and the payload are lumped with the link masses forming $m_1$ (mass on the end of the inner link) and $m_2$ (mass on the end of the outer link). The flexible inner link deforms like a Bernoulli-Euler beam. The vibration of the first link is assumed to have three degrees of freedom. An axial deformation is represented with $u$. A transverse deflection is given with $v$ and a flexible rotation of the end mass ($m_1$) is given with $\theta$. Note that when $\theta$ changes the entire rigid outer link rotates. The angle $\beta$ is the rigid motor position of the outer link relative to $m_1$. The angle $\beta$ changes with manipulator configuration but not due to vibration.

The flexible link has the three degrees of freedom that Fig. 1 labels $u$, $v$ and $\theta$. The position of the masses is given by (the values that change due to vibration are denoted as functions of time)

$$\tilde{\mathbf{r}}_1 = [l_1 - u(t)] \mathbf{i} + v(t) \mathbf{j}$$
$$\tilde{\mathbf{r}}_2 = \tilde{\mathbf{r}}_1 + l_2 \cos(\theta(t) + \beta) \mathbf{i} + l_2 \sin(\theta(t) + \beta) \mathbf{j}.$$  

By differentiating these the Kinetic energy can be determined. These terms will include the mass moments of inertia of each mass about their centroids. These inertias will be called $I_1$ and $I_2$.

The potential energy can be found by determining the strain energy generated by displacements $u$, $v$ and $\theta$. For example, the beam loading is a superposition of an axial load, a transverse load and a pure moment all applied at the free end of the beam. The axial load is independent of the other loads and is uniform along the beam therefore its strain energy is by definition

$$E_{\text{axial}} = \int_{x=0}^{l_1} \frac{AE}{2EI} \frac{\partial u^2}{\partial x} dx = \frac{1}{2} \frac{AE}{2EI} \frac{\partial u^2}{\partial x}.$$  

The values $E$ and $A$ are Young’s modulus and link cross sectional area. To determine the energies due to bending start with an expression for the bending moment $M$ at a position $x$ from the base in terms of an applied transverse load $F$ and a moment $T$

$$M = T + F(l_1 - x).$$  

Next use some elementary beam formulas [16] to express the beam’s displacement ($v$) and slope ($\theta$) at the free end in terms of applied loads. Solve these for $T$ and $F$ and substitute into the expression for moment

$$M = 2EI(2L\theta - 3v) - 6EI(-2v + L\theta)(L - x).$$
The $I$ is area moment of inertia. By definition the strain energy is the following integral:

$$ E_{\text{bending}} = \int_0^L \frac{M^2}{2EI} \, dx = 2 \left( \frac{P^2}{k_1} - 3L_\theta \theta v + 3u^2 \right) EI. $$

By adding the strain energies we arrive at the potential energy for the beam

$$ Pe = \frac{1}{2} k_{11} u(t)^2 + \frac{1}{2} \left( k_{22} v(t) - k_{23} \theta(t) \right) v(t) + \frac{1}{2} \left( k_{33} \theta(t) - k_{23} v(t) \theta(t) \right). $$

The stiffness terms are: $k_{11} = \frac{E_A}{l_1}$, $k_{22} = \frac{E_A}{l_1}$, $k_{23} = \frac{E_A}{l_1}$ and $k_{33} = \frac{4EI}{l_1}$.

According to Lagrangian mechanics, the equations of motion describing the vibration of the system are (note that the mass and stiffness matrices are symmetric)

$$ \begin{bmatrix} m_1 + m_2 & 0 & m_2 l_2 \sin(\theta(t) + \beta) \\ 0 & m_1 + m_2 & m_2 l_2 \cos(\theta(t) + \beta) \\ \cdot & \cdot & m_2 l_2 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{N F_1}{l_1} \\ \frac{N F_2}{l_1} \\ \frac{N F_3}{l_1} \end{bmatrix} = 0. \tag{1} $$

The $N F$ terms are nonlinear components composed of Coriolis and centrifugal type terms.

A dimensional analysis was performed to write the equations in nondimensional form. Define mass ratio ($\mu$)

$$ \mu = \frac{m_1}{m_2} $$

length ratio ($\gamma$)

$$ \gamma = \frac{l_1}{l_2} $$

stiffness ratio ($\kappa$)

$$ \kappa = \frac{k_{11}}{k_{22}} $$

and inertia ratio ($\iota$)

$$ \iota = \frac{l_1 + l_2}{m_2 l_1^2}. $$

Using these definitions we can write linear part (small angle $\theta$) of the original equations as

$$ \begin{bmatrix} m_2 \left[ 1 + \mu \cdot \sin(\beta) \right] & 0 & \sin(\beta) \\ 0 & 1 + \mu \cdot \cos(\beta) & \cos(\beta) \gamma^2 \mu \iota + 1 \\ \cdot & \cdot & \gamma^2 \mu \iota \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{N F_1}{l_1} \\ \frac{N F_2}{l_1} \\ \frac{N F_3}{l_1} \end{bmatrix} = 0. \tag{2} $$

If one ignores $\frac{N F_3}{l_1}$ then frequencies ($\omega$) can be determined exclusively in terms of the nondimensional terms. All results will be the nondimensional quantities $\omega$ unless otherwise stated. The eigenvalues of the system are functions of mass, length, stiffness, and inertia ratios, and relative position $\beta$.²

²The eigenvectors are also functions of these ratios and $\beta$.

The idea is that if the eigensystem changes significantly, it may be necessary to use a complicated controller to damp vibrations.

Consider the conditions that guarantee constant eigenvalues. Once the robot is designed, the mass, length, stiffness, and inertia ratios are given. Provided the payload remains constant, these ratios are constant. If the payload changes at discrete times, it might be possible to think of the system as a “set” of manipulators each with a different payload. Each of these manipulators however will still have a variable eigensystem because of its dependence on $\beta$. This behavior is illustrated in Fig. 2 that shows the first frequency of the conventional design versus relative position ($\beta$) for parameter values of $\mu = 0.1, 1, 2$, and $3$; $\gamma = 2.94$, $\iota = 0.384$, and $\kappa = 10^3$. Fig. 1 shows the vibrational motion of the conventional flexible robot. The figure shows that when the inner link vibrates, the outer link both translates and rotates. It is this rotational motion that causes the position dependent frequencies. An easy way to understand this is to compute the “equivalent” inertia of the outer link, a concept frequently used in Lagrangian mechanics. It is this concept that leads to the new design.

### III. The New Design

To see how to achieve the desirable behavior one can define a linkage as a general constraint mechanism. For example, suppose the manipulator is an open chain constructed of $n$ joints. Consider the motion of the $i$ body down the chain. Let the motion of body $i$ relative to body $i - 1$ be specified via a number of coordinates such as $\theta_i$ (motor or joint position), $\dot{\theta}_i$ (motor or joint velocity), and flexible velocity coordinates $\ddot{q}_{ij}$. If we assume $\dot{q}_{ij}$ are zero at the nominal rigid position and the linkage itself is massless then we can express the kinetic energy and potential energy introduced by linkage $i$ in terms of $\dot{\theta}_i$, $\dot{\theta}_i$, and $\dot{q}_{ij}$. Using a Lagrangian principle, we can express the dynamic equations in very general terms.

A complete investigation of the general equations is the subject of future work but one simple solution to them suggests designing a manipulator such that flexible vibration of link $i$ produces pure translation of mass $i$. One way to accomplish this is to build the flexible inner link as a parallel four-bar linkage (a PFBL). It is this solution that will be investigated in the remainder of this paper.

Fig. 3 shows the new design. In an ideal case, vibration of the inner link causes both sides of the PFBL to deform...
the same magnitude and direction causing the outer link to translate. In actuality the sides will elongate or shorten slightly causing some rotation. How much axial deformation is present determines the effectiveness of the design. The remainder of the paper discusses how close to ideal a real PFBL must behave to achieve the desired response.

A. Model of the PFBL

The PFBL link is constructed like a parallel four-bar linkage. The distance between the two flexible links measured at the base (joint assembly 1; see Fig. 3) is the same as the dimension at joint assembly 2. The rocker and the follower (side beams) are elastic and have equal length. Unlike a typical four-bar linkage, the side beams are clamped (fixed), not pinned, at the joint assembly 1 of the PFBL link. The side beams are connected via pin joints to a short rigid coupler at joint assembly 2. As in the conventional manipulator, the mass of the PFBL link is lumped at the coupler. The PFBL model has the same three degrees of freedom at joint assembly 2 as did the conventional.

1) Coupler motion transverse to the side beams measured with $v$. This is caused by common mode beam-like deformation of the side beams.

2) Coupler motion compressing the side beams measured by $u$. This is caused by common mode axial deformation of the side beams.

3) An undesirable rotation of the coupler measured by $\theta$. This is partially caused by uneven deformation of the side beams. Rotation might also result from nonparallel side beams. Nonparallel side beams might result from manufacturing tolerances or errors. The former cause of rotation is discussed in the model results, the later is discussed in a later section.

The position, velocity and hence kinetic energy in the PFBL are exactly the same as the conventional manipulator so they will not be repeated.

To compute the potential energy due to axial deformation, use a formula similar to the conventional manipulator. The only difference is the angle $\theta$ contributes to the deformation. In one beam $\theta$ adds to $u$ and in the other beam it subtracts as shown at the bottom of this page.

The bending deformation is caused exclusively by the displacement $v$ therefore it is much easier to find the strain energy

$$E_{\text{bend line}} = 2 \int_{x=0}^{h} \frac{1}{EI} \left[ 3vEI(l_1 - x) \right]^2 dx = \frac{3v^2EI}{l_1^3}.$$ 

The 2 in front of the integral is due to the fact that there are two beams bending.

According to Lagrangian mechanics, the equations of motion of the system are (remember the mass and stiffness matrices are symmetric)

$$\begin{bmatrix} m_1 + m_2 & 0 & m_2l_2\sin(\theta(t) + \beta) \\ 0 & m_1 + m_2 & m_2l_2\cos(\theta(t) + \beta) \\ 0 & 0 & m_2\beta^2 + I_1 + I_2 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = 0. \tag{3}$$

Notice that the mass matrix is the same as the conventional manipulator but the stiffness differs. There are solutions to the general equations that modify the inertia matrix but they are the subject of future work. It would not be a fair comparison to match the PFBL against the conventional if both use the same values of $A$ and $I$ in their formula. For example, using the same $A$ in the PFBL would mean it has twice the material as the conventional and one might expect it to perform better simply because of this fact. To make a more reasonable comparison, the value of $A$ and $I$ used in the PFBL will be half of the values used in the conventional. In essence we are saying that the PFBL gets to use two beams by splitting the single conventional one in half.

Again, we write the equations in nondimensional form. First introduce the aspect ratio $(A_\text{e})$

$$A_\text{e} = \frac{b}{l_1}.$$ 

Also notice that since the PFBL links have half the area moment of inertia $(I)$ as the conventional link, the value of $k_{22}$ for the PFBL will be half the value for the conventional. Let $k_{22}'$ represent the value of $k_{22}$ used in the analysis of the conventional manipulator. The equations of motion can be

$$E_{\text{axial}} = \int_{x=0}^{h_1} \frac{AE}{2l_1^2} \left( u + \frac{b\theta}{2} \right)^2 dx + \int_{x=0}^{h_1} \frac{AE}{2l_1^2} \left( u - \frac{b\theta}{2} \right)^2 dx = \frac{(b^2\theta^2 + 4u^2)AE}{4l_1}.$$
written as (again ignoring the nonlinear terms)

\[
\frac{m_2}{I_{22}} \begin{bmatrix}
1 + \mu & 0 & \sin(\beta) \\
0 & 1 + \mu & \cos(\beta) \\
\sin(\beta) & \cos(\beta) & \gamma^2 I_{11} + 1
\end{bmatrix} \begin{bmatrix}
\ddot{u} \\
\dot{v} \\
\dot{\theta}
\end{bmatrix} + \begin{bmatrix}
2\kappa & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} \gamma \kappa A_2
\end{bmatrix} \begin{bmatrix}
u \\
v \\
\theta
\end{bmatrix} = \begin{bmatrix}0 \\
0 \\
0\end{bmatrix},
\]

The results given for the PFBL will ignore the term \(\frac{m_2}{I_{22}}\). In this way the results shown for the PFBL will scale exactly the same as the results shown for the conventional manipulator. Fig. 4 shows the first frequency of the PFBL versus relative position (\(\beta\)) for parameter values of \(\mu = 0.1, 1, 2,\) and \(3; \gamma = 2.94, \lambda = 0.384,\) and \(\kappa = 10^3\).

Table I summarizes the results by showing the percent changes in frequency for the PFBL and conventional designs. The way the design works is that the stiffness corresponding to angle \(\beta\) is related to the axial stiffness of the side beams in the new design compared to the transverse stiffness of the conventional design. This increased stiffness means the magnitude of vibration in \(\beta\) should be small which means the inertia terms corresponding to \(\ddot{\theta}\) ought to be negligible. What the remainder of this paper investigates is whether the assumptions made are realistic such that a real system will exhibit the desired behavior and then to investigate the limitations of the design concept.

IV. EXPERIMENTAL RESULTS

To demonstrate that a real system will behave in the desired manner, an experiment was performed. A single aluminum PFBL (shown in Fig. 5) was constructed with an aspect ratio of approximately 0.062. The side beams were approximately 0.897 (m) long.

The experiment was performed by mounting a rigid bar on the end of the flexible arm. By placing the bar in four different orientations, the changing inertia caused by a changing \(\beta\) can be simulated. Using the same rigid beam for the four configurations maintained a constant end-mass while varying the mass-moment of inertia about the flexible arm’s tip. Fig. 6 summarizes the different end-mass mounting configurations. The moments of inertia for the rigid bar in the four different configurations are shown in Table II. The end-mass was 0.473 (kg) and 0.33 (m) long with a 0.015 (m) diameter.

A dc-response accelerometer mounted on the tip of the PFBL, a digital oscilloscope, and microcomputer were used to collect data. The robot tip was excited in a way that emphasized fundamental frequency motion. To reduce noise and high-frequency oscillation during data collection, the accelerometer had an estimated cut-off frequency of approximately 1 (Hz).

A digital fourier transform was used to compute the natural frequencies. Results of the measurements appear in Table III. The table shows that the natural frequencies are independent of the outer link orientation.

V. LIMITATIONS OF THE NEW DESIGN

A desirable response for a PFBL are specified by

1) large first natural frequency \(\omega_1\). This is because we do not want the PFBL to sacrifice rigidity;
TABLE III

RESULTS OF THE MEASUREMENTS

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Natural Frequency (Hz)</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Fig. 7. Effect of a built in non-parallelism.

2) position invariant first natural frequency. If we allow \( \beta \) to be the relative position between the inner and outer links (just as in the conventional manipulator), then position variation is defined as

\[
VR = \frac{\max_{0 \leq \beta \leq \pi} (\omega_1)}{\min_{0 \leq \beta \leq \pi} (\omega_1)} - 1
\]  

(5)

For a good design \( VR \leq \varepsilon \).

3) large separation between the first two natural frequencies. This allows the second (and higher) vibration mode(s) to be ignorable. This separation will be denoted \( \delta \) and

\[
\delta = \frac{\max_{0 \leq \beta \leq \pi} (\omega_2)}{\min_{0 \leq \beta \leq \pi} (\omega_1)}
\]

(6)

where \( \omega_k \) is the \( k \)'th lowest natural frequency.

In the following, we will investigate the effect system parameters have on these measures.

In addition to the previous, we need to consider the effect of nonparallelism. Although the manipulator is designed to be a PFBL, manufacturing tolerances when it is constructed may cause undesired behavior. Therefore the PFBL’s performance when it is slightly misaligned or nonparallel was investigated. The nonparallelism of the PFBL is quantified by factor \( \xi \) defined as a fraction of coupler width. Fig. 7 defines the terms used to compute \( \xi = \frac{b - t}{L} \). The main effect of a nonzero \( \xi \) is it introduces a rotation of the outer link. Fig. 7 also shows the rotation \( \psi \) caused by vibration and a nonzero \( \xi \). Without much trouble, the value \( \psi \) can be computed via

\[
\sin \psi = \frac{\sin(\gamma_0 + \gamma) - \sin(\gamma_0 - \gamma)}{2} = \frac{(b - t)}{L} \sin \gamma = \xi \sin \gamma.
\]

(7)

Equation (7) indicates that the transmitted angle caused by oscillation of the inner link is probably negligible because \( \xi \) will typically be a very small number.

A. Parameter Sweep Analysis

The performance of a PFBL was investigated by determining the lowest natural frequency, the frequency variation \( VR \) [see (5)] and the frequency separation \( \delta \) [see (6)] for a variety of parameters. The range of parameters was selected to be realistic values. The system parameters used in the investigation are shown in Table IV.

The mass ratio range is consistent with “payload ratios” used in previous research [17]. The payload ratio is the payload weight divided by the weight of the moving structures of a robot manipulator. For an industrial rigid robot manipulator, the payload ratio varies from \( \frac{1}{10} \) to \( \frac{1}{31} \) [18]. Since the PFBL is supposed to be a lightweight robot, we increased the range of the ratio.

In a typical robot manipulator design, the length ratio is often close to but less than one. A length ratio range from 0 to 1.5 will definitely cover all reasonable applications.

The inertia ratio, \( \iota \), is proportional to the length ratio and the mass ratio. For example, in a slender rod, the mass moment of inertia is \( \frac{m L^2}{12} \). In our case, we let the inertia ratio be \( \frac{1}{12} \).

A complete discussion of the parameter sweep results can be found elsewhere [19], here we will concentrate on a “worst case” situation. The “worst case” are parameter values from Table IV that cause the greatest \( VR \).

It is possible to define this worst case because the effect of these parameters is monotonic.

B. Design Results

What remains in our design is to determine the effect of the aspect and stiffness ratios for the PFBL manipulator. It can be shown that at small aspect ratios, the lowest frequency is approximately proportional to the aspect ratio. At zero aspect ratio, the lowest frequency is zero. The frequency quickly rises to a constant value at an aspect ratio of approximately 0.017 and stiffness ratio \( \kappa = 10^3, \gamma = 2.94, \iota = 0.384, \) and \( \mu = 3 \).

The constant frequency at first seems unusual but it results from the fact that the transverse motion \( \eta \) is not a function of the aspect ratio. This is understandable since the transverse “rocking” motion of the PFBL really has no concern for the width of the link. Also, you can see this in the mathematics. Consider the equations of motion for the PFBL [see (3)] where you can see the constant eigenvalue without much effort. It can also be shown that for the parameters given, the lowest natural frequency of the PFBL exceeds the conventional lowest frequency with \( A_{\kappa} \approx 0.015 \). This is a result of the constraining action performed by the coupler link. This means that as long as we have a reasonable aspect ratio, we need not worry about the first of our three performance criteria. The PFBL is nearly as rigid as the conventional.

Now consider the remaining two performance requirements, that is low \( VR \leq 0.1 \) and high \( \delta \geq 100 \). Consider a PFBL manipulator with \( \mu = 1.0, \gamma = 1.0, \iota = 0.1 \), and \( \kappa = 15,000 \). It can be shown that for the parameter values given, the

TABLE IV

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Parameter Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass ratio</td>
<td>( \mu )</td>
<td>[0, 3]</td>
</tr>
<tr>
<td>length ratio</td>
<td>( \gamma )</td>
<td>[0, 1.5]</td>
</tr>
<tr>
<td>inertia ratio</td>
<td>( \iota )</td>
<td>[0.05, 1.5]</td>
</tr>
<tr>
<td>stiffness ratio</td>
<td>( \kappa )</td>
<td>[10^3, 10^5]</td>
</tr>
<tr>
<td>aspect ratio</td>
<td>( A_\kappa )</td>
<td>[0.005, 1]</td>
</tr>
<tr>
<td>relative position</td>
<td>( \beta )</td>
<td>[0, ( \pi )]</td>
</tr>
</tbody>
</table>
separation between the first two frequencies monotonically increases for aspect ratios above about 0.04. We determined that separation is greater than 100 for $A_s \geq 0.31$. Similarly if $VR \leq 0.1$, then the required aspect ratio is $A_s \geq 0.03$.

The eigenvector provides information about which degree of freedom is moving at a particular vibration frequency. Although they are not shown here, it is possible to compute and plot the eigenvectors. By computing the eigenvectors associated with manipulator parameters $\mu = 1.0$, $\gamma = 1.0$, $\ell = 0.1$, and $\kappa = 15,000$, we determined that virtually all the motion at the lowest frequency is transverse rocking when the $A_s = 0.03$ and as much as 94% of the motion is transverse rocking when $A_s = 0.31$.

VI. CONCLUSION

It is possible to design a two link flexible manipulator that
1) has a nearly position invariant first natural frequency;
2) has a large separation between the first two natural frequencies;
3) and is nearly as rigid as conventional designs.

Results show that realistically sized manipulators can be constructed with the new principles. Previous work on rigid manipulators has designed the mass distribution to achieve position invariance in manipulator dynamics. This is the first work to discuss the possibility of designing the flexible behavior and to use a stiffness distribution to achieve the objective. Results for a specific example demonstrate how to select parameter values to achieve a desired behavior. Experimental results demonstrated the validity of the concept and the modeling.

For future work it is left to prove that if a manipulator has a simple intrinsic behavior, it will be easier to damp vibration with a controller. The study should consider the control complexity and robustness and determine the effect of interactions between control errors in damping vibration.

This paper establishes the idea that the open loop behavior of a flexible manipulator can be designed to be significantly different than conventional systems. Although the system configuration discussed here is simple, the concept can be extended to complex systems. The paper discussed how to derive the governing equations, their solution is the subject of future work.

REFERENCES